Econ Theory (2011) 48:125–146
DOI 10.1007/s00199-010-0542-1

RESEARCH ARTICLE

A theory of Bayesian decision making with action-dependent subjective probabilities

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Received: 11 November 2009 / Accepted: 13 May 2010 / Published online: 30 May 2010

Abstract This paper presents a complete, choice-based, axiomatic Bayesian decision theory. It introduces a new choice set consisting of information-contingent plans for choosing actions and bets and subjective expected utility model with effect-dependent utility functions and action-dependent subjective probabilities which, in conjunction with the updating of the probabilities using Bayes' rule, gives rise to a unique prior and a set of action-dependent posterior probabilities representing the decision maker's prior and posterior beliefs.

Keywords Bayesian decision making · Subjective probabilities · Prior distributions · Beliefs

JEL Classification D80 · D81 · D82

1 Introduction

Bayesian decision theory is based on the notion that a decision-maker's choice among alternative courses of action reflects his tastes for the ultimate outcomes, or payoffs, as well as his beliefs regarding the likelihoods of the events in which these payoffs materialize. The decision maker's beliefs, both prior and posterior, are supposed to be measurable cognitive phenomena quantifiable by probabilities. The essential tenets of Bayesian decision theory are two: (a) new information affects the decision maker's preferences, or choice behavior, through its effect on his beliefs rather than his tastes, and (b) the posterior probabilities, representing the decision maker's posterior beliefs, are obtained by the updating the prior probabilities, representing his prior beliefs,

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using Bayes' rule. The critical aspect of Bayesian decision theory is, therefore, the existence and uniqueness of subjective probabilities, prior and posterior, representing the decision maker's prior and posterior beliefs that abide by Bayes rule.

In the wake of the seminal work of Savage (1954), it is commonplace to depict the alternatives in the choice set as mappings from a state space, whose elements represent resolutions of uncertainty, to a set of consequences. The objects of choice have the interpretation of alternative courses of action and are referred to as acts. Much of the theory of choice consists of axiomatic models of preference relations on sets of acts whose representations involve unique subjective probabilities, interpreted as the Bayesian prior.[1] The uniqueness of the probabilities in these works is due to the use of a convention maintaining that constant acts are constant-utility acts. Lacking choice-theoretic foundations (i.e., it is not refutable in the context of the revealed-preference methodology), the use of this convention renders the prior probabilities in these models a theoretical construct that, while convenient, has no behavioral meaning. Moreover, not only does Savage's model accommodates alternative priors and corresponding (state-dependent) utility functions, the stricture that the posterior preferences be obtained from the prior preference by updating the prior probabilities according to Bayes rule, leaving the utility function intact, has no bite. Consequently, for the purpose of Bayesian updating of rankings of acts, the issue of uniqueness of the probabilities has no empirical relevance. However, form the point of view of Bayesian statistics, the non-uniqueness of the prior is a fundamental flaw. Note, in particular, that rather than choosing state-independent utility function, it is possible normalize the utilities so that the prior be uniform. Hence, every Bayesian analysis may always start from a uniform prior regardless of the decision maker's beliefs.

In this paper, I propose an alternative analytical framework and a behavioral model that characterizes a subjective expected utility representation of the decision-maker's preferences, involving a unique family of action-dependent priors on effects, and corresponding families of action-dependent posteriors. In addition, the utility functions that figure in the representation may be effect-dependent, the significance of which is discussed below. This work extends the analytical framework of Karni (2006) by including, in addition to actions, effects, and bets, observations, and strategies. Actions are initiatives by which decision makers believe they can affect the likely realization of effects. Effects are observable realizations of eventualities on which decision makers can place bets, and which might also be of direct concern to them. Bets are real-valued mapping on the set of effects. Observations correspond to informative signals that the decision maker may receive before choosing his action and bet. Strategies are maps from the set of observations to the set of action-bet pairs. In this model, decision makers are characterized by preference relations on the set of all strategies whose axiomatic structure lends the notion of constant utility bets choice-theoretic meaning. In this model it possible to define a unique family of action-dependent, joint subjective probability distributions on the product set of effects and observations. Moreover, the prior probabilities are the unconditional marginal probabilities on the set of effects

[1] Prominent among these theories are the subjective expected utility models of Savage (1954), Anscombe and Aumann (1963), and Wakker (1989), and the probability sophisticated choice models of Machina and Schmeidler (1992, 1995).

and the posterior probabilities are the distributions on the effects conditional on the observations. Finally, most importantly, these prior and posterior probabilities are the only representations of the decision maker's prior and posterior beliefs that are consistent with the tenets of Bayesian decision model mentioned above.

This issue here is not purely theoretical. Karni (2008b) gives an example involving the design of optimal insurance in the presence of moral hazard, in which the insurer knows the insured's prior preferences and assumes, correctly, that the insured is Bayesian. The example shows that, failure to ascribe to the insured his true prior probabilities and utilities may result in attributing to him the wrong posterior preferences. In such case, when new information (for instance, a study indicating a decline in the incidence of theft in the neighborhood in which the insured resides) necessitates changing the terms of the insurance policy, the insurer may offer the insured a policy that is individually rational and incentive incompatible. More generally, in the presence of moral hazard, correct prediction of an agent's changing behavior by the application of Bayes rule requires that the agent be ascribed a prior that faithfully represents his beliefs. A more meaningful notion of subjective probability, one that is a measurement of subjective beliefs when these beliefs have structure that allows their representation by probability measure, is developed in this paper.

As indicated above, this paper extends that of Karni (2006) by including two new ingredients, namely, observations and strategies, that, together with the actions, make it possible to identify constant utility bets. The latter are essential for a choice-based definition of Bayesian priors that do not relay on arbitrary normalization of the utility function. More specifically, Karni (2006) introduced the notion of constant valuation bets. Unlike constant-utility bets, constant-valuation bets are defined using compensating variations between the direct utility cost associated with the actions and their impact on the probabilities of the effects. The uniqueness of the probability in Karni (2006), must still rely on an arbitrary normalization of the utility functions. Providing a choice-based definition of constant utility bets and, thereby, ridding the model of the need for an arbitrary normalization of the utility functions, is one the main aspects of this paper. In addition, in Karni (2006) the direct utility impact of the actions enters the representation through action-dependent transformations of the expected-utility functionals. By contrast, in the present model of these transformations appear as additive terms thus rendering the representation simpler and consistent with the modeling of agents' actions in the literature on incentive contracts.

The model presented here accommodates effect-dependent preferences, lending itself to natural interpretations in the context of medical decision making and the analysis of life insurance, health insurance, as well as standard portfolio and property insurance problems. The fact that the probabilities are action dependent means that the model furnishes an axiomatic foundation for the behavior of the principal and agent depicted in the parametrized distribution formulation of agency theory introduced by Mirrlees (1974, 1976).[2]

[2] The axiomatic foundations of agency theory was first explored in Karni (2008a). That study invoked the analytical framework developed in Karni (2007), in which the choice set consists of action-bet pairs and the payoffs of the bets are lotteries. That framework included neither observations nor strategies. Consequently, the notion of constant utility bets, which is central to the present work, was impossible to define. Instead,

The pioneering attempt to extend the subjective expected utility model to include moral hazard and state-dependent preferences is due to Drèze (1961, 1987). Invoking the analytical framework of Anscombe and Aumann (1963), he departed from their "reversal of order" axiom, assuming instead that decision makers may strictly prefer knowing the outcome of a lottery before the state of nature becomes known. According to Drèze, this suggests that the decision maker believes that he can influence the probabilities of the states. How this influence is produced is not made explicit. The representation entails the maximization of subjective expected utility over a convex set of subjective probability measures.[3]

The next section introduces the theory and the main results. Concluding remarks appear in Sect. 3. The proof of the main representation theorem appears in Sect. 4.

2 The theory

2.1 The analytical framework

Let   be a finite set of *effects*, X a finite set of *observations* or *signals,* and A a connected separable topological space whose elements are referred to as *actions*. Actions correspond to initiatives (e.g., time and effort) that decision makers may take to influence the likely realization of effects.

A *bet* is a real-valued mapping on   interpreted as monetary payoffs contingent on the realization of the effects. Let B denote the set of all bets on   and assume that it is endowed with the $\mathbb{R}^{|\ \ |}$ topology. Denote by $(b_{-\theta}r)$ the bet obtained from $b \in B$ by replacing the θ coordinate of b (that is, $b(\theta)$) with r. Effects are analogous to Savage (1954) states in the sense that they resolve the uncertainty associated with the payoff of the bets. Unlike states, however, the likely realization of effects may conceivably be affected by the decision maker's actions.[4]

Observations may be obtained before the choice of bets and actions, in which case they affect these choices. For example, upon learning the result of a new study concerning the effect of cholesterol level in blood on the likelihood of a heart attack, a decision maker may adopt an exercise and diet regimen to reduce the risk of heart attack and, at the same time, take out health insurance and life insurance policies. In this instance the new findings correspond to observations, the diet and exercise regimen correspond to actions, the states of health are effects, and the financial terms of an insurance policy constitute a bet on .[5]

Footnote 2 continued
constant valuation bets in conjunction with an arbitrary normalization of the utility functions were used together to pin down action dependent subjective probabilities and outcome-dependent utilities of money.

[3] The model in this paper differs from that of Drèze in several important respects, including the specification of the means by which a decision maker thinks he may influence the likelihood of the alternative effects. For more details see Karni (2006).

[4] It is sufficient, for my purpose, that the decision maker believes that he may affect the likely realization of the effects by his choice of action.

[5] Clearly, the information afforded by the new observation is conditioned by the existing regimen. The decision problem is how to modify the existing regimen in light of the new information.

To model this "dynamic" aspect of the decision making process, I assume that a decision maker formulates a strategy, or contingent plan, specifying the action-bet pairs to be implemented contingent on the observations. Formally, denote by o the event "no new information" and let $\bar{X} = X \cup \{o\}$, then a *strategy* is a function $I : \bar{X} \to A \times B$ that has the interpretation of a set of instructions specifying, for each $x \in \bar{X}$, an action-bet pair to be implemented if x is observed.[6] Let $\mathcal{I}$ be the set of all strategies.

A decision maker is characterized by a preference relation on $\mathcal{I}$. The strict preference relation, , and the indifference relation, $\sim$, are the asymmetric and symmetric parts of , respectively. Denote by $I_{-x}(a, b) \in \mathcal{I}$ the strategy in which the x coordinate of I is replaced by (a, b) . An observation, x, is *essential* if $I_{-x}(a, b)$ $I_{-x}\left(a , b \right)$ for some (a, b) , $\left(a , b \right) \in A \times B$ and $I \in \mathcal{I}$. I assume throughout that all elements of $\bar{X}$ are essential.

In the terminology of Savage (1954), $\bar{X}$ may be interpreted as a set of states and contingent plans as acts. However, because the decision maker's beliefs about the likelihoods of the effects depend on both the actions and the observations, the preferences on action-bet pairs are inherently observation dependent. Thus applying Savage's state-independent axioms, P3 and P4, to on $\mathcal{I}$, makes no sense.

To grasp the role of the various ingredients of the model and set the stage for the statement of the axioms, it is useful, at this junction, to look ahead at the representation of on $\mathcal{I}$. The representation involves an array of continuous, effect-dependent utility functions $\{u(\cdot, \theta) : \mathbb{R} \to \mathbb{R}\}_{\theta \in}$ and a utility of actions function $v : A \to \mathbb{R}$ unique up to common positive linear transformation, and a unique family of action-dependent joint probability measures, $\{\pi(\cdot, \cdot \mid a)\}_{a \in A}$ on $\bar{X} \times$ such that on $\mathcal{I}$ is represented by

$$I \to \sum_{x \in \bar{X}} \sum_{\theta \in} \pi\left(x, \theta \mid a_{I(x)}\right) \left[u\left(b_{I(x)}(\theta), \theta\right) + v\left(a_{I(x)}\right)\right], \tag{1}$$

where $b_{I(x)}$ and $a_{I(x)}$ are the bet and action assigned by the strategy I to the observation x. Furthermore, for all $x \in \bar{X}$, $\mu(x) := \sum_{\theta \in} \pi(x, \theta \mid a)$ is independent of a. Hence the representation (1) may be written as

$$I \to \sum_{x \in \bar{X}} \mu(x) \left[\sum_{\theta \in} \pi\left(\theta \mid x, a_{I(x)}\right) u\left(b_{I(x)}(\theta), \theta\right) + v\left(a_{I(x)}\right) \right., \tag{2}$$

where, for all $x \in X$, $\pi(\theta \mid x, a) = \pi(x, \theta \mid a) / \mu(x)$ is the posterior probability of θ conditional on x and a, and for each $a \in A$, $\pi(\theta \mid o, a) = \frac{1}{1-\mu(o)} \sum_{x \in X} \pi(x, \theta \mid a)$ is the prior probability of θ conditional a.[7]

[6] Alternatively stated, o is a non-informative observation (that is, anticipating the representation below, the subjective probability distribution on the effects conditional on o is the same as that under the current information).

[7] Describing $\pi(\cdot \mid o, a)$ as the prior distribution is appropriate because conditioning on o is means that not information is obtained before a decision is taken.

In either representation the choice of strategy entails evaluation of the bets by their expected utility. Actions enter this representation as a direct source of (dis)utility as well as an instrument by which the decision maker believes he may affect the likely realizations of the effects.

2.2 Axioms and additive representation of on $\mathcal{I}$

The first axiom is standard:

(A.1) (**Weak order**) *is a complete and transitive binary relation.*

A topology on $\mathcal{I}$ is needed to define continuity of the preference relation . Recall that $\mathcal{I} = (A \times B)^{\bar{X}}$ and let $\mathcal{I}$ be endowed with the product topology.[8]

(A.2) (**Continuity**) *For all* $I \in \mathcal{I}$, *the sets* $\{I \ \in \mathcal{I} \mid I \quad I\}$ *and* $\{I \ \in \mathcal{I} \mid I \quad I \ \}$ *are closed.*

The next axiom, coordinate independence, is analogous to but weaker than Savage (1954) sure thing principle.[9]

(A.3) (**Coordinate independence**) *For all* $x \in \bar{X}$, $I, I \ \in \mathcal{I}$, *and* $(a, b), (a \ , b \ ) \in A \times B$, $I_{-x}(a, b) \quad I_{-x}(a, b)$ *if and only if* $I_{-x}(a \ , b \ ) \quad I_{-x}(a \ , b \ )$.

An array of real-valued functions $(v_s)_{s \in S}$ is said to be a *jointly cardinal additive representation* for a binary relation on a product set $D = \ _{s \in S} D_s$ if, for all $d, d \ \in D$, $d \quad d$ if and only if $\sum_{s \in S} v_s(d_s) \geq \sum_{s \in S} v_s(d_s)$, and the class of all functions that constitute an additive representation of consists of those arrays of functions, $(\hat{v}_s)_{s \in S}$, for which $\hat{v}_s = \eta v_s + \zeta_s$, $\eta > 0$ for all $s \in S$. The representation is continuous if the functions v_s, $s \in S$ are continuous.

The following theorem is an application of Theorem III.4.1 in Wakker (1989)[10]:

Theorem 1 *Let* $\mathcal{I}$ *be endowed with the product topology and* $|\bar{X}| \geq 3$. *Then a preference relation on* $\mathcal{I}$ *satisfies* (A.1)–(A.3) *if and only if there exist an array of real-valued functions* $\{w(\cdot, \cdot, x) \mid x \in \bar{X}\}$ *on* $A \times B$ *that constitute a jointly cardinal, continuous, additive representation for* .

2.3 Independent betting preferences

For every given $x \in \bar{X}$, denote by ^{x} the induced preference relation on $A \times B$ defined by $(a, b) \ ^x \ (a \ , b \ )$ if and only if $I_{-x}(a, b) \quad I_{-x}(a \ , b \ )$. The induced strict preference relation, denoted by ^{x}, and the induced indifference relation, denoted by $\sim^x$, are the asymmetric and symmetric parts of ^{x}, respectively.[11] The induced preference relation ^{o} is referred to as the *prior* preference relation; the preference relations

[8] That is, the topology on $\mathcal{I}$ is the product topology on the Cartesian product $(A \times B)^{|\bar{X}|}$.

[9] See Wakker (1989) for details.

[10] To simplify the exposition I state the theorem for the case in which $\bar{X}$ contains at least three essential coordinates. Additive representation when there are only two essential coordinates requires the imposition of the hexagon condition (see Wakker 1989 theorem III.4.1).

[11] For preference relations satisfying (A.1)–(A.3), these relations are well-defined.

^{x}, $x \in X$, are the *posterior* preference relations. For each $a \in A$ the preference relation $\ ^{x}$ induces a conditional preference relation on B defined as follows: for all $b, b \in B$, $b \ _{a}^{x}\ b$ if and only if $(a, b) \ ^{x}\ (a, b)$. The asymmetric and symmetric part of $\ _{a}^{x}$ are denoted by $\ _{a}^{x}$ and $\sim_{a}^{x}$, respectively.

An effect, θ, is said to be *nonnull given the observation–action pair* (x, a) if $(b_{-\theta}r) \ _{a}^{x}\ (b_{-\theta}r)$, for some $b \in B$ and $r, r \in \mathbb{R}$; it is *null given the observation-action pair* (x, a) otherwise. Given a preference relation, , denote by (a, x) the subset of effects that are nonnull given the observation–action pair (x, a). Assume that $(a, o) = \Theta$, for all $a \in A$.

Two effects, θ and θ , are said to be *elementarily linked* if there are actions $a, a \in A$ and observations $x, x \in \bar{X}$ such that $\theta, \theta \in \ (a, x) \cap \ (a, x)$. Two effects are said to be *linked* if there exists a sequence of effects $\theta = \theta_0, \ldots, \theta_n = \theta$ such that θ_j and θ_{j+1} are elementarily linked, $j = 0, \ldots, n-1$. The set of effects, Θ, is linked if every pair of its elements is linked.

The next axiom requires that the "intensity of preferences" for monetary payoffs contingent on any given effect be independent of the action and the observation:

(A.4) (**Independent betting preferences**) *For all* $(a, x), (a, x) \in A \times \bar{X}$, $b, b, b, b \in B$, $\theta \in \ (a, x) \cap \ (a, x)$, *and* $r, r, r, r \in \mathbb{R}$, *if* $(b_{-\theta}r) \ _{a}^{x}\ (b_{-\theta}r)$, $(b_{-\theta}r) \ _{a}^{x}\ (b_{-\theta}r)$, *and* $(b_{-\theta}r) \ _{a}^{x}\ (b_{-\theta}r)$ *then* $(b_{-\theta}r) \ _{a}^{x}\ (b_{-\theta}r)$.

To grasp the meaning of independent betting preferences, think of the preferences $(b_{-\theta}, r) \ _{a}^{x}\ (b_{-\theta}, r)$ and $(b_{-\theta}, r) \ _{a}^{x}\ (b_{-\theta}, r)$ as indicating that given the action a, the observation x, and the effect θ, the intensity of the preferences of r over r is sufficiently larger than that of r over r as to reverse the preference ordering of the effect-contingent payoffs $b_{-\theta}$ and $b_{-\theta}$. The axiom requires that these intensities not be contradicted when the action is a instead of a and the observation is x instead of x.

The idea may be easier to grasp by considering a specific instance in which $(b_{-\theta}, r) \sim_{a}^{x} (b_{-\theta}, r)$, $(b_{-\theta}r) \sim_{a}^{x} (b_{-\theta}r)$ and $(b_{-\theta}r) \sim_{a}^{x} (b_{-\theta}r)$. The first pair of indifferences indicates that, given a and x, the difference in the payoffs b and b contingent on the effects other than θ measures the intensity of preferences between the payoffs r and r and between r and r, contingent on θ. The indifference $(b_{-\theta}r) \sim_{a}^{x} (b_{-\theta}r)$ then indicates that given another action–observation pair, a and x, the intensity of preferences between the payoffs r and r contingent on θ is measured by the difference in the payoffs the bets b and b contingent on the effects other than θ. The axiom requires that, in this case, the difference in the payoffs b and b contingent on the effects other than θ is also a measure of the intensity of the payoffs r and r contingent on θ. Thus the intensity of preferences between two payoffs given θ is independent of the actions and the observations.

2.4 Belief consistency

To link the decision maker's prior and posterior probabilities, the next axiom asserts that for every $a \in A$ and $\theta \in \Theta$, the prior probability of θ given a is the sum over X

of the joint probability distribution on $X \times$ conditional on θ and a (that is, the prior is the marginal probability on).

Let $I^{-o}(a, b)$ denote the strategy that assigns the action–bet pair (a, b) to every observation other than o (that is, $I^{-o}(a, b)$ is a strategy such that $I(x) = (a, b)$ for all $x \in X$).

(A.5) (**Belief consistency**) *For every* $a \in A$, $I \in \mathcal{I}$ *and* $b, b \in B$, $I_{-o}(a, b) \sim I_{-o}(a, b)$ *if and only if* $I^{-o}(a, b) \sim I^{-o}(a, b)$.

The interpretation of Axiom (A.5) is as follows. The decision maker is indifferent between two strategies that agree on X and, in the event that no new information becomes available, call for the implementation of the alternative action–bet pairs (a, b) or (a, b) if and only if he is indifferent between two strategies that agree on o and call for the implementation of the same action–bet pairs (a, b) or (a, b) regardless of the observation. Put differently, given any action, the preferences on bets conditional on there being no new information is the same as that when new information may not be used to select the bet. Hence, in and of itself, information is worthless.

2.5 Constant utility bets

Constant utility bets are bets whose payoffs offset the direct impact of the effects. Formally

Definition 2 A bet $\bar{b} \in B$ is a constant utility bet according to if, for all $I, I, I, I \in \mathcal{I}$, $a, a, a, a \in A$ and $x, x \in \bar{X}$, $I_{-x}(a, \bar{b}) \sim I_{-x}(a, \bar{b})$, $I_{-x}(a, \bar{b}) \sim I_{-x}(a, \bar{b})$ and $I_{-x}(a, \bar{b}) \sim I_{-x}(a, \bar{b})$ imply $I_{-x}(a, \bar{b}) \sim I_{-x}(a, \bar{b})$ and $\cap_{(x,a)\in X\times A}\{b \in B \mid b \sim_a^x \bar{b}\} = \{\bar{b}\}$.

To render the definition meaningful it is assumed that, given $\bar{b}$, for all a, a, $a, a \in A$ and $x, x \in \bar{X}$ there are $I, I, I, I \in \mathcal{I}$ such that the indifferences $I_{-x}(a, \bar{b}) \sim I_{-x}(a, \bar{b})$, $I_{-x}(a, \bar{b}) \sim I_{-x}(a, \bar{b})$ and $I_{-x}(a, \bar{b}) \sim I_{-x}(a, \bar{b})$ hold.

As in the interpretation of axiom (A.4), to understand the definition of constant utility bets it is useful to think of the preferences $I_{-x}(a, \bar{b}) \sim I_{-x}(a, \bar{b})$ and $I_{-x}(a, \bar{b}) \sim I_{-x}(a, \bar{b})$ as indicating that, given $\bar{b}$ and x, the preferential difference between the substrategies I_{-x} and I_{-x} measure the intensity of preference of a over a and that of a over a. The indifference $I_{-x}(a, \bar{b}) \sim I_{-x}(a, \bar{b})$ implies that, given $\bar{b}$, and another observation x, the preferential difference between the substrategies I_{-x} and I_{-x} is another measure the intensity of preference of a over a. Then it must be true that it also measure the intensity of preference of a over a.

The requirement that $\cap_{(x,a)\in X\times A}\{b \in B \mid b \sim_a^x \bar{b}\} = \{\bar{b}\}$ implicitly asserts that actions and observations affect the probabilities of the effects, and that these actions and observations are sufficiently rich so that $\bar{b}$ is well-defined. It is worth emphasizing that the axiomatic structure does not rule out that the decision maker believes that his choice of action does not affect the likelihoods of the effects. However, the uniqueness part of Definition 2, by excluding the existence of distinct constant utility bets belonging to the same equivalence classes, for all $(a, x) \in A \times X$, implies that, not only does the decision maker believe in his ability to affect the likely realization of the effects by his choice of action, but also that these likelihoods depend on the observations.

To understand why this implies that $\bar{b}$ is a constant utility bet recall that, in general, actions affect decision makers in two ways: directly through their utility cost and indirectly by altering the probabilities of the effects. Moreover, only the indirect impact depends on the observations. The definition requires that, given $\bar{b}$, the intensity of the preferences over the actions be observation-independent. This means that the indirect influence of the actions is neutralized, which can happen only if the utility associated with $\bar{b}$ is invariable across the effects.

Let $B^{cu}(\ )$ be a subset of all constant utility bets according to . In general, this set may be empty. This is the case if the range of the utility of the monetary payoffs across effects do not overlap. Here I am concerned with the case in which $B^{cu}(\ )$ is nonempty. The set $B^{cu}(\ )$ is said to be *inclusive* if for every $(x, a) \in X \times A$ and $b \in B$ there is $\bar{b} \in B^{cu}(\ )$ such that $b \sim^{x}_{a} \bar{b}$.[12]

The next axiom requires that the trade-offs between the actions and the substrategies that figure in Definition 2 are independent of the constant utility bets.

(A.6) (**Trade-off independence**) For all $I, I \in \mathcal{I}$, $x \in \bar{X}$, $a, a \in A$ and $\bar{b}, \bar{b} \in B^{cu}(\ )$, $I_{-x}\left(a, \bar{b}\right) \quad I_{-x}\left(a, \bar{b}\right)$ if and only if $I_{-x}\left(a, \bar{b}\right) \quad I_{-x}\left(a, \bar{b}\right)$.

Finally, it is also required that the direct effect (that is, cost) of actions, measured by the preferential difference between $\bar{b}$ and $\bar{b}$ in $B^{cu}(\ )$, be independent of the observation.

(A.7) (**Conditional monotonicity**) For all $\bar{b}, \bar{b} \in B^{cu}(\ )$, $x, x \in \bar{X}$, and $a, a \in A$, $\left(a, \bar{b}\right) \ ^{x} \left(a, \bar{b}\right)$ if and only if $\left(a, \bar{b}\right) \ ^{x} \left(a, \bar{b}\right)$.

2.6 The main representation theorem

The next theorem asserts the existence of subjective expected utility representation of the preference relation on $\mathcal{I}$, and characterizes the uniqueness properties of its constituent utilities and the probabilities. For each $I \in \mathcal{I}$ let $\left(a_{I(x)}, b_{I(x)}\right)$ denote the action-bet pair corresponding to the x coordinate of I, i.e., $I(x) = \left(a_{I(x)}, b_{I(x)}\right)$.

Theorem 3 *Let be a preference relation on $\mathcal{I}$ and suppose that $B^{cu}(\ )$ is inclusive, then:*

(a) *The following two conditions are equivalent:*

(a.i) *satisfies* (A.1)–(A.7)

(a.ii) *there exist continuous, real-valued functions* $\{u(\cdot, \theta) \mid \theta \in \ \}$ *on* R, $v \in R^{A}$, *and a family,* $\{\pi(\cdot, \cdot \mid a) \mid a \in A\}$, *of joint probability measures on* $\bar{X} \times$ *such that on* I *is represented by*

$$I \to \sum_{x \in \bar{X}} \mu(x) \left[\sum_{\theta \in} \pi\left(\theta \mid x, a_{I(x)}\right) u\left(b_{I(x)}(\theta), \theta\right) + v\left(a_{I(x)}\right) \right., \tag{3}$$

[12] Inclusiveness of $B^{cu}(\ )$ simplifies the exposition. For existence and uniqueness of the probabilities in the main result below it is enough that for every given x and a, $B^{cu}(\ )$ contains at least two bets.

where $\mu(x) = \sum_{\theta \in} \pi(x, \theta \mid a)$ *for all* $x \in \bar{X}$ *is independent of* a, $\pi(\theta \mid x, a) = \pi\left(x, \theta \mid a\right) / \mu(x)$ *for all* $(x, a) \in \bar{X} \times A$, $\pi\left(\theta \mid o, a\right) = \frac{1}{1-\mu(o)} \sum_{x \in X} \pi\left(x, \theta \mid a\right)$ *for all* $a \in A$, *and, for every* $\bar{b} \in B^{cu}(\quad)$, $u\left(\bar{b}(\theta), \theta\right) = u\left(\bar{b}\right)$, *for all* $\theta \in \quad$.

(b) *If* $\{\hat{u}(\cdot, \theta) \mid \theta \in \quad\}$, $\hat{v} \in R^A$ and $\{\hat{\pi}(\cdot, \cdot \mid a) \mid a \in A\}$ *is another set of utilities and a family of joint probability measures representing* *in the sense of* (3), *then* $\hat{\pi}(\cdot, \cdot \mid a) = \pi(\cdot, \cdot \mid a)$ *for every* $a \in A$ *and there are numbers* $m > 0$ *and* k, k *such that* $\hat{u}(\cdot, \theta) = m\hat{u}(\cdot, \theta) + k$, $\theta \in \quad$ *and* $\hat{v} = mv + k$.

Although the joint probability distributions $\pi(\cdot, \cdot \mid a), a \in A$ depend on the actions, the distribution μ is independent of a. This is consistent with the formulation of the decision problem according to which the choice of actions is contingent on the observations. In other words, if new information in the form of an observation becomes available, it precedes the choice of action. Consequently, the dependence of the joint probability distributions $\pi(\cdot, \cdot \mid a)$ on a captures solely the decision maker's beliefs about his ability to influence the likelihood of the effects by his choice of action.[13]

The key to obtaining the uniqueness of the joint probability distributions $\pi(\cdot, \cdot \mid a)$, $a \in A$ is the existence and uniqueness of constant utility bets. The definition of these bets requires, in turn, that the decision maker perceives the likelihoods of the effects to depends on both his actions and the observations. It is worth underscoring that, neither actions nor observations can be dispense with and still obtain a choice-based definition of constant utility bets.

Unlike the subjective probability in the theory of Savage (1954) (and in all other theories that invoke Savage's analytical framework) whose uniqueness is predicated on an arbitrary specification of the utility function, the uniqueness of the probabilities in this theory is entirely choice based. In particular, the theory of this paper is immune to the critique of Savage's theory in the introduction.

3 Concluding remarks

3.1 Effect-independent preferences and effect-independent utility functions

The choice-based Bayesian decision theory presented in this paper includes, as a special case, effect-independent preferences. In particular, following Karni (2006), effect independent preferences is captured by the following axiom:

(A.8) **Effect-independent betting preferences** *For all* $x \in \bar{X}$, $a \in A$, $b, b, b, b \in B$, $\theta, \theta \in \Theta$, *and* $r, r, r\ r \in \mathbb{R}$, *if* $(b_{-\theta}, r) \;{}^{x}_{a}\; (b_{-\theta}, r)$, $(b_{-\theta}, r) \;{}^{x}_{a}\; (b_{-\theta}, r)$, *and* $(b_{-\theta}, r) \;{}^{x}_{a}\; (b_{-\theta}, r)$ *then* $(b_{-\theta}, r) \;{}^{x}_{a}\; (b_{-\theta}, r)$.

[13] If an action-effect pair are already "in effect" when new information arrives, they constitute a default course of action. In such instance, the interpretation of the decision at hand is possible choice of new action and bet. For example, a modification of a diet regimen coupled with a possible change of life insurance policy.

The interpretation of this axiom is analogous to that of action-independent betting preferences. The preferences $(b_{-\theta}, r\ )\ \ {}^{x}_{a}\ (b_{-\theta}, r)$ and $(b_{-\theta}, r\ )\ \ {}^{x}_{a}\ (b_{-\theta}, r\ )$ indicate that, for every given (a, x), the "intensity" of the preference for r over r given the effect θ is sufficiently greater than that of r over r as to reverse the order of preference between the payoffs $b_{-\theta}$ and $b_{-\theta}$. Effect independence requires that these intensities not be contradicted by the preferences between the same payoffs given any other effect θ.

Adding axiom (A.8) to the hypothesis of Theorem 3 implies that the utility function that figures in the representation takes the form $u\left(b\left(\theta\right), \theta\right) = t\left(\theta\right) u\left(b\left(\theta\right)\right) + s\left(\theta\right)$, where $t\left(\theta\right) > 0$. In other words, even if the preference relation exhibits *effect-independence* over bets, the *utility function* may still display effect dependence, in the form of the additive and multiplicative coefficient. Thus, effects may impact the decision maker's well-being without necessarily affecting his risk preferences.

Let B^c be the subset of constant bets (that is, trivial bets with the same payoff regardless of the effect that obtains). If the set of constant utility bets coincides with the set of constant bets (that is, $B^c = B^{cu}\ (\quad)$), then the utility function is effect independent (that is, $u\left(b\left(\theta\right), \theta\right) = u\left(b\left(\theta\right)\right)$ for all $\theta \in \Theta$). The implicit assumption that the set of constant utility bets coincides with the set of constant bets is the convention invoked by the standard subjective utility models. Unlike in those models, however, in the theory of this paper, this assumption is a testable hypothesis.

3.2 Conditional preferences and dynamic consistency

The specification of the decision problem implies that, before the decision maker chooses an action-bet pair, either no informative signal arrives (that is, the observation is o) or new informative signal arrives in the form of an observation $x \in X$. One way or another, given the information at his disposal, the decision maker must choose among action-bet pairs. Let $\left(\hat{\ }^{x}\right)_{x\in\bar{X}}$ be binary relations on $A \times B$ depicting the decision maker's choice behavior conditional on observing x. I refer to $\left(\hat{\ }^{x}\right)_{x\in\bar{X}}$ by the name *ex-post preference relations*.

Dynamic consistency requires that at each $x \in \bar{X}$, the decision maker implements his plan of action envisioned for that contingency by the original strategy. Formally,

Definition 4 A preference relation on $\mathcal{I}$ is dynamically consistent with the ex-post preference relations $\left(\hat{\ }^{x}\right)_{x\in\bar{X}}$ on $A \times B$ if the posterior preference relations $\left(\ \ ^{x}\right)_{x\in\bar{X}}$ satisfy $\ ^{x} = \hat{\ }^{x}$ for all $x \in \bar{X}$.

The following is an immediate implication of Theorem 3.

Corollary 5 *Let be preference relation on $\mathcal{I}$ satisfying* (A.1)–(A.7) *and suppose that $B^{cu}\ (\quad)$ is inclusive. Then is dynamically consistent with the ex-post preference relations $\left(\hat{\ }^{x}\right)_{x\in\bar{X}}$ on $A \times B$ if and only if, for all $x \in \bar{X}$, $\hat{\ }^{x}$ is represented by*

$$(a, b) \to \sum_{\theta\in} \pi\left(\theta \mid x, a\right) u\left(b\left(\theta\right), \theta\right) + v\left(a\right), \tag{4}$$

where $\{u(\cdot,\theta) \mid \theta \in \quad\}$ *and* $\{\pi(\cdot \mid x, a) \mid x \in \bar{X}, a \in A\}$ *are the utility functions and conditional subjective probabilities that appear in the representation* (3).

For every $a \in A$ the subjective action-contingent prior on is $\pi(\cdot \mid o, a)$ and the subjective action-contingent posteriors on are $\pi(\cdot \mid x, a)$, $x \in X$. The subjective action-dependent prior is the marginal distribution on induced by the distribution on $X \times \Theta$, and the subjective action-dependent posteriors are obtained from the action-contingent joint distribution on $X \times$ by conditioning on the observation.

4 Proof of Theorem 3

For expository convenience, I write B^{cu} instead of $B^{cu}(\quad)$.

(a) $(a.i) \Rightarrow (a.ii)$. Suppose that on $\mathcal{I}$ satisfies (A.1)–(A.7) and B^{cu} is inclusive. By Theorem 1, is represented by

$$I \to \sum_{x \in \bar{X}} w\left(a_{I(x)}, b_{I(x)}, x\right). \tag{5}$$

where $w(.,.,x)$, $x \in \bar{X}$ are jointly cardinal, continuous, real-valued functions.

Since satisfies (A.4), Lemmas 4 and 5 in Karni (2006) applied to ^{x}, $x \in \bar{X}$, and Theorem III.4.1 in Wakker (1989) imply that for every $(a, x) \in A \times \bar{X}$ such that (a, x) contains at least two effects, there exist array of functions $\{v_{(a,x)}(\cdot;\theta) : \mathbb{R} \to \mathbb{R} \mid \theta \in \quad\}$ that constitute a jointly cardinal, continuous additive representation of $^{x}_{a}$ on B. Moreover, by the proof of Lemma 6 in Karni (2006), satisfies (A.1)–(A.4) if and only if, for every (a, x), $\left(a', x'\right) \in A \times \bar{X}$ such that $(a, x) \cap \left(a', x'\right) = \varnothing$ and $\theta \in (a, x) \cap \left(a', x'\right)$, there exist $\beta_{((a',x'),(a,x),\theta)} > 0$ and $\alpha_{((a',x'),(a,x),\theta)}$ satisfying $v_{(a',x')}(\cdot,\theta) = \beta_{((a',x'),(a,x),\theta)} v_{(a,x)}(\cdot,\theta) + \alpha_{((a',x'),(a,x),\theta)}$.[14]

Fix $\hat{a} \in A$ and define $u(\cdot,\theta) = v_{(\hat{a},o)}(\cdot,\theta)$, $\lambda(a, x;\theta) = \beta_{((a,x),(\hat{a},o),\theta)}$ and $\alpha(a, x, \theta) = \alpha_{((a,x),(\hat{a},o),\theta)}$ for all $a \in A$, $x \in \bar{X}$, and $\theta \in \Theta$. For every given $(a, x) \in A \times \bar{X}$, $w(a, b, x)$ represents $^{x}_{a}$ on B. Hence

$$w(a, b, x) = H\left(\sum_{\theta \in} (\lambda(a, x, \theta)\, u(b(\theta);\theta) + \alpha(a, x, \theta)), a, x\right), \tag{6}$$

where H is a continuous, increasing function.

Consider next the restriction of to $(A \times B^{cu})^{\bar{X}}$.

Lemma 6 *There exist a function* $U : A \times B^{cu} \to \mathbb{R}$, $\xi \in \mathbb{R}^{|\bar{X}|}_{++}$ *and* $\zeta \in \mathbb{R}^{|\bar{X}|}$ *such that, for all* $\left(a, \bar{b}, x\right) \in A \times B^{cu} \times \bar{X}$,

$$w\left(a, \bar{b}, x\right) = \xi(x)\, U\left(\bar{b}, a\right) + \zeta(x). \tag{7}$$

[14] By definition, for all (a, x) and θ, $\beta_{((a,x),(a,x),\theta)} = 1$ and $\alpha_{((a,x),(a,x),\theta)} = 0$.

Proof Let $I, I, I, I \in \mathcal{I}$, $a, a, a, a \in A$ and $\bar{b}$ be as in Definition 2. Then, for all $x, x \in \bar{X}$, $I_{-x}\left(a, \bar{b}\right) \sim I_{-x}\left(a, \bar{b}\right)$, $I_{-x}\left(a, \bar{b}\right) \sim I_{-x}\left(a, \bar{b}\right)$, $I_{-x}\left(a, \bar{b}\right) \sim I_{-x}\left(a, \bar{b}\right)$ and $I_{-x}\left(a, \bar{b}\right) \sim I_{-x}\left(a, \bar{b}\right)$. By the representation (5), $I_{-x}\left(a, \bar{b}\right) \sim I_{-x}\left(a, \bar{b}\right)$ implies that

$$\sum_{y \in \bar{X}-\{x\}} w\left(a_{I(y)}, b_{I(y)}, y\right) + w\left(a, \bar{b}, x\right) = \sum_{y \in \bar{X}-\{x\}} w\left(a_{I(y)}, b_{I(y)}, y\right) + w\left(a, \bar{b}, x\right). \tag{8}$$

Similarly, $I_{-x}\left(a, \bar{b}\right) \sim I_{-x}\left(a, \bar{b}\right)$ implies that

$$\sum_{y \in \bar{X}-\{x\}} w\left(a_{I(y)}, b_{I(y)}, y\right) + w\left(a, \bar{b}, x\right) = \sum_{y \in \bar{X}-\{x\}} w\left(a_{I(y)}, b_{I(y)}, y\right) + w\left(a, \bar{b}, x\right), \tag{9}$$

$I_{-x}\left(a, \bar{b}\right) \sim I_{-x}\left(a, \bar{b}\right)$ implies that

$$\begin{aligned} &\sum_{y \in \bar{X}-\{x\}} w\left(a_{I(y)}, b_{I(y)}, y\right) + w\left(a, \bar{b}, x\right) \\ &\quad = \sum_{y \in \bar{X}-\{x\}} w\left(a_{I(y)}, b_{I(y)}, y\right) + w\left(a, \bar{b}, x\right), \end{aligned} \tag{10}$$

and $I_{-x}\left(a, \bar{b}\right) \sim I_{-x}\left(a, \bar{b}\right)$ implies that

$$\begin{aligned} &\sum_{y \in \bar{X}-\{x\}} w\left(a_{I(y)}, b_{I(y)}, y\right) + w\left(a, \bar{b}, x\right) \\ &\quad = \sum_{y \in \bar{X}-\{x\}} w\left(a_{I(y)}, b_{I(y)}, y\right) + w\left(a, \bar{b}, x\right). \end{aligned} \tag{11}$$

But (8) and (9) imply that

$$w\left(a, \bar{b}, x\right) - w\left(a, \bar{b}, x\right) = w\left(a, \bar{b}, x\right) - w\left(a, \bar{b}, x\right). \tag{12}$$

and (10) and (11) imply that

$$w\left(a, \bar{b}, x\right) - w\left(a, \bar{b}, x\right) = w\left(a, \bar{b}, x\right) - w\left(a, \bar{b}, x\right). \tag{13}$$

Define a function $\phi_{(x,x,\bar{b})}$ as follows: $w\left(\cdot, \bar{b}, x\right) = \phi_{(x,x,\bar{b})} \circ w\left(\cdot, \bar{b}, x\right)$. Axiom (A.7) with $\bar{b} = \bar{b}$ imply that it is monotonic increasing. Then $\phi_{(x,x,\bar{b})}$ is continuous. Moreover, (12) and (13) in conjunction with Lemma 4.4 in Wakker (1987) imply that $\phi_{(x,x,\bar{b})}$ is affine.

Let $\beta_{(x,o,\bar{b})} > 0$ and $\delta_{(x,o,\bar{b})}$ denote, respectively, the multiplicative and additive coefficients corresponding to $\phi_{(x,o,\bar{b})}$, where the inequality follows from the monotonicity of $\phi_{(x,o,\bar{b})}$. Observe that, by (A.6), $I_{-o}\left(a, \bar{b}\right) \sim I_{-o}\left(a\ , \bar{b}\right)$ if and only if $I_{-o}\left(a, \bar{b}\ \right) \sim I_{-o}\left(a\ , \bar{b}\ \right)$. Hence

$$\beta_{(x,o,\bar{b})}\left[w\left(a, \bar{b}, o\right) - w\left(a\ , \bar{b}, o\right)\right] = \beta_{(x,o,\bar{b}\ )}\left[w\left(a, \bar{b}\ , o\right) - w\left(a\ , \bar{b}\ , o\right)\right] \quad (14)$$

for all $\bar{b}, \bar{b}\ \in B^{cu}$. Thus, for all $x \in \bar{X}$ and $\bar{b}, \bar{b}\ \in B^{cu}$, $\beta_{(x,o,\bar{b})} = \beta_{(x,o,\bar{b}\ )} := \xi(x) > 0$.

Let $a, a\ \in A$ and $\bar{b}, \bar{b}\ \in B^{cu}$ satisfy $\left(a, \bar{b}\right) \sim^{o} \left(a\ , \bar{b}\ \right)$. By axiom (A.7) $\left(a, \bar{b}\right) \sim^{o} \left(a\ , \bar{b}\ \right)$ if and only if $\left(a, \bar{b}\right) \sim^{o} \left(a\ , \bar{b}\ \right)$. By the representation this equivalence implies that

$$w\left(a, \bar{b}, o\right) = w\left(a\ , \bar{b}\ , o\right). \quad (15)$$

if and only if,

$$\xi(x)\, w\left(a, \bar{b}, o\right) + \delta_{(x,o,\bar{b})} = \xi(x)\, w\left(a\ , \bar{b}\ , o\right) + \delta_{(x,o,\bar{b}\ )}. \quad (16)$$

Thus $\delta_{(x,o,\bar{b})} = \delta_{(x,o,\bar{b}\ )}$.

By this argument and continuity (A.2) the conclusion can be extended to B^{cu}. Let $\delta_{(x,o,\bar{b})} := \zeta(x)$ for all $\bar{b} \in B^{cu}$.

For every given $\bar{b} \in B^{cu}$ and all $a \in A$, define $U\left(\bar{b}, a\right) = w\left(a, \bar{b}, o\right)$. Then, for all $x \in \bar{X}$,

$$w\left(a, \bar{b}, x\right) = \xi(x)\, U\left(\bar{b}, a\right) + \zeta(x),\ \xi(x) > 0. \quad (17)$$

This completes the proof of Lemma 6.

Equations (6) and (7) imply that for every $x \in \bar{X}, \bar{b} \in B^{cu}$ and $a \in A$,

$$\xi(x)\, U\left(\bar{b}, a\right) + \zeta(x) = H\quad \sum_{\theta \in} \lambda(a, x, \theta)\, u\left(\bar{b}(\theta), \theta\right) + \hat{\alpha}(a, x), a, x\quad . \quad (18)$$

Lemma 7 *The identity* (18) *holds if and only if* $u\left(\bar{b}(\theta), \theta\right) = u\left(\bar{b}\right)$ *for all* $\theta \in \Theta$, $\sum_{\theta \in}\ \frac{\lambda(a,x,\theta)}{\xi(x)} = \varphi(a)$, $\frac{\hat{\alpha}(a,x)}{\xi(x)} = v(a)$ *for all* $a \in A$,

$$H\quad \sum_{\theta \in} \lambda(a, x, \theta)\, u\left(\bar{b}(\theta), \theta\right) + \hat{\alpha}(a, x), a, x\quad = \xi(x)\left[u\left(\bar{b}\right) + v(a)\right] + \zeta(x), \quad (19)$$

and there is $\kappa(a) > 0$ *such that*

$$\kappa(a) \sum_{\theta \in} \frac{\lambda(a, x, \theta)}{\xi(x)} u\left(\bar{b}(\theta), \theta\right) + \frac{\hat{\alpha}(a, x)}{\xi(x)} = U\left(\bar{b}, a\right). \tag{20}$$

Proof (Sufficiency) Let $u\left(\bar{b}(\theta), \theta\right) := u\left(\bar{b}\right)$ for all $\theta \in \Theta$, $\sum_{\theta \in} \frac{\lambda(a,x,\theta)}{\xi(x)} := \varphi(a)$ and $c(a) := \kappa(a)\varphi(a)$ for all $a \in A$ and suppose that (20) holds.

But axiom (A.6) and the representation imply that, for all $\bar{b}, \bar{b} \in B^{cu}$,

$$c(a)\, u\left(\bar{b}\right) + v(a) = c\left(a\right) u\left(\bar{b}\right) + v\left(a\right)$$

if and only if

$$c(a)\, u\left(\bar{b}\right) + v(a) = c\left(a\right) u\left(\bar{b}\right) + v\left(a\right).$$

Hence $c(a) = c\left(a\right) = c$ for all $a, a \in A$.

Normalize u so that $c = 1$. Then Eq. (18) follows from Eqs. (19) and (20).

(Necessity) Multiply and divide the first argument of H by $\xi(x) > 0$. Equation (18) may be written as follows:

$$\xi(x)\, U\left(\bar{b}, a\right) + \zeta(x) = H \quad \xi(x)\left[\sum_{\theta \in} \frac{\lambda(a, x, \theta)}{\xi(x)} u\left(\bar{b}(\theta), \theta\right) + \frac{\hat{\alpha}(a, x)}{\xi(x)} \quad , a, x \quad . \tag{21}$$

Define $V\left(a, \bar{b}, x\right) = \sum_{\theta \in} \frac{\lambda(a,x,\theta)}{\xi(x)} u\left(\bar{b}(\theta), \theta\right) + \frac{\hat{\alpha}(a,x)}{\xi(x)}$ then, for every given $(a, x) \in A \times X$ and all $\bar{b}, \bar{b} \in B^{cu}$,

$$\begin{aligned} U\left(\bar{b}, a\right) - U\left(\bar{b}, a\right) = \big[&H\left(\xi(x)\, V\left(a, \bar{b}, x\right), a, x\right) \\ &- H\left(\xi(x)\, V\left(a, \bar{b}, x\right), a, x\right)\big] / \xi(x). \end{aligned} \tag{22}$$

Hence $H(\cdot, a, x)$ is a linear function whose intercept is $\zeta(x)$ and the slope

$$\left[U\left(\bar{b}, a\right) - U\left(\bar{b}, a\right)\right] / \left[V\left(a, \bar{b}, x\right) - V\left(a, \bar{b}, x\right)\right] := \kappa(a),$$

is independent of x. Thus

$$\xi(x)\, U\left(\bar{b}, a\right) + \zeta(x) = \kappa(a)\, \xi(x)\left[\sum_{\theta \in} \frac{\lambda(a, x, \theta)}{\xi(x)} u\left(\bar{b}(\theta), \theta\right) + \frac{\hat{\alpha}(a, x)}{\xi(x)} \quad + \zeta(x). \tag{23}$$

Hence

$$U\left(\bar{b}, a\right) / \kappa(a) = \sum_{\theta \in} \frac{\lambda(a, x, \theta)}{\xi(x)} u\left(\bar{b}(\theta), \theta\right) + \frac{\hat{\alpha}(a, x)}{\xi(x)} \tag{24}$$

is independent of x. However, because $\overset{x}{\underset{a}{=}}\succeq_{a}^{x}$ for all a and some $x, x \in \bar{X}$, in general, $\lambda(a, x, \theta)/\xi(x)$ is not independent of θ. Moreover, because $\hat{\alpha}(a, x)/\xi(x)$ is independent of b, the first term on the right-hand side of (24) must be independent of x. For this to be true $u\left(\bar{b}(\theta), \theta\right)$ must be independent of θ and $\sum_{\theta\in} \lambda(a, x, \theta)/\xi(x) := \varphi(a)$ be independent of x. Moreover, because the first term on the right-hand side of (24) is independent of x, $\hat{\alpha}(a, x)/\xi(x)$ must also be independent of x. Finally, by definition, $\bar{b}$ the unique element in its equivalence class that has the property that $u\left(\bar{b}(\theta), \theta\right)$ is independent of θ.

Define $v(a) := \hat{\alpha}(a, x)/\xi(x)$, $u\left(\bar{b}(\theta), \theta\right) = u\left(\bar{b}\right)$, for all $\theta \in \Theta$, and $U\left(\bar{b}, a\right) = u\left(\bar{b}\right) + v(a)$ and $\kappa(a)\varphi(a) = 1$. Thus

$$U\left(\bar{b}, a\right) = \kappa(a) \sum_{\theta\in} \frac{\lambda(a, x, \theta)}{\xi(x)} u\left(\bar{b}(\theta); \theta\right) + \frac{\hat{\alpha}(a, x)}{\xi(x)}. \tag{25}$$

This completes the proof of Lemma 7.

Note that

$$U\left(\bar{b}, a\right) = \sum_{\theta\in} \frac{\lambda(a, x, \theta)}{\xi(x)\varphi(a)} u\left(\bar{b}(\theta); \theta\right) + \frac{\hat{\alpha}(a, x)}{\xi(x)} = u\left(\bar{b}\right) + v(a). \tag{26}$$

But, by Lemma 7, $\sum_{\theta\in} \lambda(a, x, \theta) = \xi(x)\varphi(a)$. Hence, by the inclusivity of B^{cu}, the representation (5) is equivalent to

$$I \to \sum_{x\in\bar{X}} \left[\sum_{\theta\in} \frac{\lambda\left(a_{I(x)}, x, \theta\right)}{\sum_{\theta\in} \lambda\left(a_{I(x)}, x, \theta\right)} u\left(b_{I(x)}(\theta); \theta\right) + \frac{\hat{\alpha}\left(a_{I(x)}, x\right)}{\xi(x)} \right. . \tag{27}$$

For all $x \in X$, $a \in A$ and $\theta \in \Theta$, define the joint subjective probability distribution on $\times \bar{X}$ by

$$\pi(x, \theta \mid a) = \frac{\lambda(a, x, \theta)}{\sum_{x \in\bar{X}} \sum_{\theta \in} \lambda(a, x, \theta)}. \tag{28}$$

Since $\sum_{\theta\in} \lambda(a, x, \theta) = \xi(x)\varphi(a)$, for all $x \in \bar{X}$,

$$\sum_{\theta\in} \pi(x, \theta \mid a) = \frac{\xi(x)\varphi(a)}{\sum_{x \in\bar{X}} \xi(x)\varphi(a)} = \frac{\xi(x)}{\sum_{x \in\bar{X}} \xi(x)}. \tag{29}$$

Define the subjective probability of $x \in \bar{X}$ as follows:

$$\mu(x) = \frac{\xi(x)}{\sum_{x \in\bar{X}} \xi(x)}. \tag{30}$$

Then the subjective probability of x is given by the marginal distribution on X induced by the joint distributions $\pi(\cdot, \cdot \mid a)$ on $X \times$ and is independent of a.

For all $\theta \in \Theta$, define the subjective posterior and prior probability of θ, respectively, by

$$\pi(\theta \mid x, a) = \frac{\pi(x, \theta \mid a)}{\mu(x)} = \frac{\lambda(a, x, \theta)}{\sum_{\theta \in} \lambda(a, x, \theta)} \tag{31}$$

and

$$\pi(\theta \mid o, a) = \frac{\lambda(a, o, \theta)}{\sum_{\theta \in} \lambda(a, o, \theta)}. \tag{32}$$

Substitute in (27) to obtain the representation (3),

$$I \to \sum_{x \in \bar{X}} \mu(x) \left[\sum_{\theta \in} \pi\left(\theta \mid x, a_{I(x)}\right) u\left(b_{I(x)}(\theta), \theta\right) + v\left(a_{I(x)}\right) \right. . \tag{33}$$

Let $a \in A$, $I \in \mathcal{I}$ *and* $b, b \in B$, satisfy $I_{-o}(a, b) \sim I_{-o}\left(a, b\right)$. Then, by (33),

$$\sum_{\theta \in} \pi(\theta \mid o, a) u(b(\theta), \theta) = \sum_{\theta \in} \pi(\theta \mid o, a) u\left(b\ (\theta), \theta\right) \tag{34}$$

and, by axiom (A.5) and (33)

$$\sum_{x \in X} \frac{\mu(x)}{1 - \mu(0)} \sum_{\theta \in} \pi(\theta | x, a) u(b(\theta), \theta) = \sum_{x \in X} \frac{\mu(x)}{1 - \mu(0)} \sum_{\theta \in} \pi(\theta \mid x, a) u\left(b\ (\theta), \theta\right). \tag{35}$$

Thus

$$\sum_{\theta \in} \left[u(b(\theta), \theta) - u\left(b\ (\theta), \theta\right)\right] \left[\pi(\theta \mid o, a) - \sum_{x \in X} \frac{\mu(x)}{1 - \mu(0)} \pi(\theta \mid x, a) \right. = 0. \tag{36}$$

This implies that $\pi(\theta \mid o, a) = \sum_{x \in X} \mu(x)\pi(\theta \mid x, a)/[1 - \mu(0)]$.

(If $\pi(\theta \mid o, a) > \sum_{x \in X} \mu(x)\pi(\theta \mid x, a)/[1 - \mu(0)]$ for some θ and $\mu(o)\pi(\theta \mid o, a) < \sum_{x \in X} \mu(x)\pi(\theta \ \mid x, a)/[1 - \mu(0)]$ for some θ , let $\hat{b}, \hat{b} \ \in B$ be such that $\hat{b}(\theta) > b(\theta)$ and $\hat{b}(\hat{\theta}) = b(\hat{\theta})$ for all $\hat{\theta} \in \quad - \{\theta\}$, $\hat{b}\ (\theta\ ) > b\ (\theta\ )$ and $\hat{b}\ (\hat{\theta}) = b\ (\hat{\theta})$ for all $\hat{\theta} \in \quad - \{\theta\ \}$ and $I_{-o}(a, \hat{b}) \sim I_{-o}(a, \hat{b}\ )$. Then

$$\sum_{\theta \in} u \quad \hat{b}(\theta), \theta\Big) - u \quad \hat{b}\ (\theta), \theta\Big)\Big] \left[\pi(\theta \mid o, a) - \sum_{x \in X} \frac{\mu(x)}{1 - \mu(0)} \pi(\theta \mid x, a) \right. > 0. \tag{37}$$

But this contradicts (A.5).)

$(a.ii) \Rightarrow (a.i)$. The necessity of (A.1), (A.2) and (A.3) follows from Theorem 1. To see the necessity of (A.4), suppose that $I_{-x}(a, b_{-\theta} r) \quad I_{-x}\left(a, b_{-\theta} r\right)$, $I_{-x}\left(a, b_{-\theta} r\right) \quad I_{-x}\left(a, b_{-\theta} r\right)$, and $I_{-x}\left(a, b_{-\theta} r\right) \quad I_{-x}\left(a, b_{-\theta} r\right)$. By representation (6)

$$\sum_{\theta \in \quad -\{\theta\}} \lambda\left(a, x, \theta\right) u\left(b\left(\theta\right), \theta\right) + \lambda\left(a, x, \theta\right) u\left(r, \theta\right) \geq \sum_{\theta \in \quad -\{\theta\}} \lambda\left(a, x, \theta\right) u\left(b\left(\theta\right), \theta\right) + \lambda\left(a, x, \theta\right) u\left(r, \theta\right), \tag{38}$$

$$\sum_{\theta \in \quad -\{\theta\}} \lambda\left(a, x, \theta\right) u\left(b\left(\theta\right), \theta\right) + \lambda\left(a, x, \theta\right) u\left(r, \theta\right) \geq \sum_{\theta \in \quad -\{\theta\}} \lambda\left(a, x, \theta\right) u\left(b\left(\theta\right), \theta\right) + \lambda\left(a, x, \theta\right) u\left(r, \theta\right), \tag{39}$$

and

$$\sum_{\theta \in \quad -\{\theta\}} \lambda\left(a, x, \theta\right) u\left(b\left(\theta\right), \theta\right) + \lambda\left(a, x, \theta\right) u\left(r, \theta\right) \geq \sum_{\theta \in \quad -\{\theta\}} \lambda\left(a, x, \theta\right) u\left(b\left(\theta\right), \theta\right) + \lambda\left(a, x, \theta\right) u\left(r, \theta\right). \tag{40}$$

But (38) and (39) imply that

$$u\left(r, \theta\right) - u\left(r, \theta\right) \geq \frac{\sum_{\theta \in \quad -\{\theta\}} \lambda\left(a, x, \theta\right)\left[u\left(b\left(\theta\right), \theta\right) - u\left(b\left(\theta\right), \theta\right)\right]}{\lambda\left(a, x, \theta\right)} \geq u\left(r, \theta\right) - u\left(r, \theta\right). \tag{41}$$

Inequality (40) implies

$$u\left(r, \theta\right) - u\left(r, \theta\right) \geq \frac{\sum_{\theta \in \quad -\{\theta\}} \lambda\left(a, x, \theta\right)\left[u\left(b\left(\theta\right), \theta\right) - u\left(b\left(\theta\right), \theta\right)\right]}{\lambda\left(a, x, \theta\right)}. \tag{42}$$

But (41) and (42) imply that

$$u\left(r, \theta\right) - u\left(r, \theta\right) \geq \frac{\sum_{\theta \in \quad -\{\theta\}} \lambda\left(a, x, \theta\right)\left[u\left(b\left(\theta\right), \theta\right) - u\left(b\left(\theta\right), \theta\right)\right]}{\lambda\left(a, x, \theta\right)}. \tag{43}$$

Hence

$$\sum_{\theta \in \ -\{\theta\}} \lambda\left(a , x , \theta \right)\left[u\left(b \ \ (\theta), \theta \right) - u\left(b \ \ (\theta), \theta \right)\right]$$
$$+ \lambda\left(a , x , \theta\right)\left[u\left(r \ , \theta\right) - u\left(r \ \ , \theta\right)\right] \geq 0. \tag{44}$$

Thus, $I_{-x} \ \left(a , b_{-\theta} r \ \right) \quad I_{-x} \ \left(a , b_{-\theta} r \ \ \right)$.

Next I show that if $\bar{b} \in B$ satisfies $u\left(\bar{b}(\theta), \theta\right) = u\left(\bar{b}\right)$ for all $\theta \in$ then $\bar{b} \in B^{cu}$. Suppose that representation (3) holds and let $I, I , I \ , I \quad \in \mathcal{I}$, $a, a , a \ , a \quad \in A$ and $x, x \ \in \bar{X}$, such that $I_{-x}\left(a, \bar{b}\right) \sim I_{-x}\left(a , \bar{b}\right)$, $I_{-x}\left(a \ , \bar{b}\right) \sim I_{-x}\left(a \ \ , \bar{b}\right)$ and $I_{-x} \ \left(a , \bar{b}\right) \sim I_{-x} \ \left(a, \bar{b}\right)$. Then the representation (5) implies that

$$\sum_{\hat{x} \in \bar{X}-\{x\}} w \ \ a_{I(\hat{x})}, b_{I(\hat{x})}, \hat{x}\Big) + \mu(x)\left[u\left(\bar{b}\right) + v(a)\right]$$
$$= \sum_{\hat{x} \in \bar{X}-\{x\}} w \ \ a_{I \ (\hat{x})}, b_{I \ (\hat{x})}, \hat{x}\Big) + \mu(x)\left[u\left(\bar{b}\right) + v\left(a \ \right)\right] \tag{45}$$

$$\sum_{\hat{x} \in \bar{X}-\{x\}} w \ \ a_{I \ (\hat{x})}, b_{I \ (\hat{x})}, \hat{x}\Big) + \mu(x)\left[u\left(\bar{b}\right) + v\left(a \ \ \right)\right]$$
$$= \sum_{\hat{x} \in \bar{X}-\{x\}} w \ \ a_{I(\hat{x})}, b_{I(\hat{x})}, \hat{x}\Big) + \mu(x)\left[u\left(\bar{b}\right) + v\left(a \ \ \ \right)\right] \tag{46}$$

and

$$\sum_{\hat{x} \in \bar{X}-\{x \ \}} w \ \ a_{I \ \ (\hat{x})}, b_{I \ \ (\hat{x})}, \hat{x}\Big) + \mu\left(x \ \right)\left[u\left(\bar{b}\right) + v\left(a \ \right)\right]$$
$$= \sum_{\hat{x} \in \bar{X}-\{x \ \}} w \ \ a_{I \ \ (\hat{x})}, b_{I \ \ (\hat{x})}, \hat{x}\Big) + \mu\left(x \ \right)\left[u\left(\bar{b}\right) + v(a)\right]. \tag{47}$$

But (45) and (46) imply that

$$v(a) - v\left(a \ \right) = v\left(a \ \ \right) - v\left(a \ \ \ \right). \tag{48}$$

Equality (47) implies

$$\frac{\sum_{\hat{x} \in \bar{X}-\{x \ \}} \ w \ \ a_{I \ \ (\hat{x})}, b_{I \ \ (\hat{x})}, \hat{x}\Big) - w \ \ a_{I \ \ (\hat{x})}, b_{I \ \ (\hat{x})}, \hat{x}\Big)\Big]}{\mu\left(x \ \right)} = v(a) - v\left(a \ \right). \tag{49}$$

Thus

$$\sum_{\hat{x} \in \bar{X}-\{x \ \}} w \ \ a_{I \ \ (\hat{x})}, b_{I \ \ (\hat{x})}, \hat{x}\Big) + u\left(\bar{b}\right) + v\left(a \ \ \right) = \sum_{\hat{x} \in \bar{X}-\{x \ \}} w \ \ a_{I \ \ (\hat{x})}, b_{I \ \ (\hat{x})}, \hat{x}\Big)$$
$$+ u\left(\bar{b}\right) + v\left(a \ \ \right) \tag{50}$$

Hence $I_{-x}\left(a\ \ ,\bar{b}\right)\sim I_{-x}\left(a\ \ ,\bar{b}\right)$ and $\bar{b}\in B^{cu}$.

To show the necessity of (A.5) let $a\in A$, $I\ \in\mathcal{I}$ and $b,b\ \ \in B$, by the representation $I_{-o}\left(a,b\right)\sim I_{-o}\left(a,b\ \right)$ if and only if

$$\sum_{\theta\in}\pi\left(\theta\mid o,a\right)u\left(b\left(\theta\right),\theta\right)=\sum_{\theta\in}\pi\left(\theta\mid o,a\right)u\left(b\ \ \left(\theta\right),\theta\right). \tag{51}$$

But $\pi\left(\theta\mid o,a\right)=\sum_{x\in X}\mu\left(x\right)\pi\left(\theta\mid x,a\right)/\left[1-\mu\left(0\right)\right]$. Thus (51) holds if and only if

$$\sum_{x\in X}\mu\left(x\right)\sum_{\theta\in}\pi\left(\theta\mid x,a\right)u\left(b\left(\theta\right),\theta\right)=\sum_{x\in X}\mu\left(x\right)\sum_{\theta\in}\pi\left(\theta\mid x,a\right)u\left(b\ \ \left(\theta\right),\theta\right). \tag{52}$$

But (52) is valid if and only if $I^{-o}\left(a,b\right)\sim I^{-o}\left(a,b\ \right)$.

For all I and x, let $K(I,x)=\sum_{y\in X-\{x\}}\mu\left(y\right)\left[\sum_{\theta\in}\ \pi\left(\theta\mid x,a\right)u\left(b_{I\left(y\right)}\left(\theta\right)\right)+v\left(a_{I\left(y\right)}\right)\right]$. To show the necessity of (A.6) Then $I_{-x}\left(a,\bar{b}\right)\quad I_{-x}\left(a\ ,\bar{b}\right)$ if and only if

$$K(I,x)+u\left(\bar{b}\right)+v\left(a\right)\geq K(I\ ,x)+u\left(\bar{b}\right)+v\left(a\ \right) \tag{53}$$

if and only if

$$K(I,x)+u\left(\bar{b}\ \right)+v\left(a\right)\geq K(I\ ,x)+u\left(\bar{b}\ \right)+v\left(a\ \right) \tag{54}$$

if and only if $I_{-x}\left(a,\bar{b}\ \right)\quad I_{-x}\left(a\ ,\bar{b}\ \right)$.

To show that axiom (A.7) is implied, not that $I_{-x}\left(a,\bar{b}\right)\quad I_{-x}\left(a\ ,\bar{b}\ \right)$ if and only if

$$K(I,x)+u\left(\bar{b}\right)+v\left(a\right)\geq K(I,x)+u\left(\bar{b}\ \right)+v\left(a\ \right) \tag{55}$$

if and only if

$$K(I,x\ )+u\left(\bar{b}\right)+v\left(a\right)\geq K(I,x\ )+u\left(\bar{b}\ \right)+v\left(a\ \right) \tag{56}$$

if and only if $I_{-x}\left(a,\bar{b}\right)\quad I_{-x}\left(a\ ,\bar{b}\ \right)$. This completes the proof of part (a).

(b) Suppose, by way of negation, that there exist continuous, real-valued functions $\{\tilde{u}\left(\cdot,\theta\right)\mid\theta\in\quad\}$ on $\mathbb{R}$, $\tilde{v}\in\mathbb{R}^{A}$ and, for every $a\in A$, there is a joint probability measure $\tilde{\pi}\left(\cdot,\cdot\mid a\right)$ on $\bar{X}\times\Theta$, distinct from those that figure in the representation (3), such that on $\mathcal{I}$ is represented by

$$I\rightarrow\sum_{x\in\bar{X}}\tilde{\mu}\left(x\right)\left[\sum_{\theta\in}\tilde{\pi}\left(\theta\mid x,a_{I(x)}\right)\tilde{u}\left(b_{I(x)}\left(\theta\right),\theta\right)+\tilde{v}\left(a_{I(x)}\right)\quad,\right. \tag{57}$$

where $\tilde{\mu}(x) = \sum_{\theta \in} \tilde{\pi}(x, \theta \mid a)$ for all $x \in \bar{X}$, and $\tilde{\pi}(\theta \mid x, a) = \tilde{\pi}(x, \theta \mid a) / \tilde{\mu}(x)$ for all $(\theta, x, a) \in \quad \times X \times A$.

Define $\kappa(x) = \tilde{\mu}(x) / \mu(x)$, for all $x \in \bar{X}$. Then the representation (57) may be written as

$$I \to \sum_{x \in \bar{X}} \mu(x) \left[\sum_{\theta \in} \pi\left(\theta \mid x, a_{I(x)}\right) \gamma\left(\theta, x, a_{I(x)}\right) \kappa(x) \tilde{u}\left(b_{I(x)}(\theta), \theta\right) \right.$$

$$\left. + \kappa(x) \tilde{v}\left(a_{I(x)}\right) \right. . \tag{58}$$

Hence, by (3), $\tilde{u}(b(\theta), \theta) = u(b(\theta), \theta) / \tilde{\gamma}(\theta, x, a) \kappa(x)$ and $\tilde{v}(a) = v(a) / \kappa(x)$. The second equality implies that $\kappa(x) = \kappa$ for all $x \in \bar{X}$. Consequently, the first inequality implies that $\tilde{\gamma}(\theta, x, a) = \gamma(\theta)$ for all $(x, a) \in \bar{X} \times A$. Thus, for $\bar{b} \in B^{cu}$,

$$I \to \sum_{x \in \bar{X}} \mu(x) \left[\sum_{\theta \in} \pi\left(\theta \mid x, a_{I(x)}\right) \frac{u\left(\bar{b}\right)}{\gamma(\theta)} + v\left(a_{I(x)}\right) \right. . \tag{59}$$

Let $\hat{b} \in B$ be defined by $u \quad \hat{b}(\theta), \theta\Big) = u\left(\bar{b}\right) / \gamma(\theta)$ for all $\theta \in \Theta$. Then, $\hat{b} \sim_a^x \bar{b}$ for all $(x, a) \in \bar{X} \times A$, and, by Definition 2, $\hat{b} \in B^{cu}$. Moreover, if $\gamma(\cdot)$ is not a constant function then $\hat{b} = \bar{b}$. This contradicts the uniqueness of $\bar{b}$ in Definition 2. Thus $\gamma(\theta) = \gamma$ for all $\theta \in \Theta$. But

$$1 = \sum_{x \in \bar{X}} \sum_{\theta \in} \tilde{\pi}\left(\theta, x \mid a_{I(x)}\right) = \gamma \sum_{x \in \bar{X}} \sum_{\theta \in} \pi(\theta, x \mid a) = \gamma. \tag{60}$$

Hence, $\tilde{\pi}(\theta, x \mid a) = \pi(\theta, x \mid a)$ for all $(\theta, x) \in \quad \times \bar{X}$ and $a \in A$.

Next consider the uniqueness of the utility functions. The representations (3) and (5) imply that

$$w(a, b, x) = \mu(x) \left[\sum_{\theta \in} \pi(\theta \mid x, a) u(b(\theta), \theta) + v(a) \right. . \tag{61}$$

Hence, by the uniqueness part of Theorem 1, $\{\tilde{u}(\cdot, \theta)\}_{\theta \in}$ and $\tilde{v} \in R^A$ must satisfy

$$\sum_{\theta \in} \pi(\theta \mid x, a) v(b(\theta), \theta) + \tilde{v}(a) = m \left[\sum_{\theta \in} \pi(\theta \mid x, a) u(b(\theta), \theta) + v(a) \right.$$

$$+ K(x), \tag{62}$$

where $m > 0$. Clearly, this is the case if $\tilde{u}(\cdot, \theta) = mu(\cdot, \theta) + k$ and $\tilde{v} = mv + k$.

Suppose that $\tilde{u}(\cdot, \theta) = mu(\cdot, \theta) + k$, $\tilde{v} = m \; v + k$ and, without loss of generality, let $m > m > 0$. Take $a, a \in A$ and $\bar{b}, \bar{b} \in B^{cu}$ such that $\left(a, \bar{b}\right) \sim^x \left(a , \bar{b} \right)$. Then, by the representation (3),

$$u\left(\bar{b}\right) - u\left(\bar{b}\ \right) = v\,(a) - v\left(a\ \right). \tag{63}$$

But

$$\tilde{u}\left(\bar{b}\right) - \tilde{u}\left(\bar{b}\ \right) = m\left[u\left(\bar{b}\right) - u\left(\bar{b}\ \right)\right] > m\ \left[v\,(a) - v\left(a\ \right)\right] = \tilde{v}\,(a) - \tilde{v}\left(a\ \right). \tag{64}$$

Hence $\tilde{u}\,(\cdot,\theta)$ and $\tilde{v}$ do not represent .

Consider next $\tilde{u}\,(\cdot,\theta) = mu\,(\cdot,\theta) + k\,(\theta)$, where $k\,(\cdot)$ is not a constant function. Let $\bar{k}\,(x,a) = \sum_{\theta\in}\ \pi\,(\theta \mid x,a)\,k\,(\theta)$. Take $a, a\ \in A$ and $\bar{b}, \bar{b}\ \in B^{cu}$ such that $\left(a,\bar{b}\right) \sim^{x} \left(a\ ,\bar{b}\ \right)$ and $\left[\bar{k}\,(x,a) - \bar{k}\left(x,a\ \right)\right] = 0$ for some x. Then

$$\begin{aligned}\tilde{u}\left(\bar{b}\right) - \tilde{u}\left(\bar{b}\ \right) &= m\left[u\left(\bar{b}\right) - u\left(\bar{b}\ \right)\right] + \left[\bar{k}\,(x,a) - \bar{k}\left(x,a\ \right)\right] = m\left[v\,(a) - v\left(a\ \right)\right]\\ &= \tilde{v}\,(a) - \tilde{v}\left(a\ \right).\end{aligned}$$

Hence $\tilde{u}\,(\cdot,\theta)$ and $\tilde{v}$ do not represent .

If $\tilde{v}\,(a) = mv\,(a) + k\ \ (a)$, where $k\ \ (\cdot)$ is not a constant function then, by a similar argument, $\tilde{u}\,(\cdot,\theta)$ and $\tilde{v}$ do not represent .

Acknowledgments I am grateful to Tsogbadral Galaabaatar, Jacques Drèze, Robert Nau and an anonymous referee for their useful comments and suggestions.

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